Last time: - operator exponential
- Banach spaces
- segue into Hilbert space

Let V be an inner product space.

Def: The map $V \to \mathbb{R}$

$$v \longmapsto \sqrt{\langle v, v \rangle}$$

is called the 2-norm or Euclidean norm on V

Thm: The 2-norm is a norm.

So any inner product space is naturally a normed space. So we can give it the norm topology.

Def: A Hilbert space is an inner product space which is (Cauchy) complete using the Euclidean norm.

The results from last class show that

Thm: Any finite dimensional inner product space is a Hilbert space.

Def: A subspace of a normed space which is a closed set is called a closed subspace.

Recall that for any subset S of a topological space, the closure of S is the set of all adherent points of S, denoted $\bar{S}$.

Problem: Prove that if S is a subspace of a normed space V, then $\bar{S}$ is also a subspace V.

Lemma: In a normed space V, if S is any set, then a vector $v \in V$ is an adherent point of S iff v is the limit of a sequence of points in S.
Furthermore, we have

a) $\lim_n (v_n + w_n) = \lim_n v_n + \lim_n w_n$

b) $\lim_n (\lambda v_n) = \lambda \lim_n v_n$.

Def: A subset S of a topological space X is **dense** if $\bar{S} = X$.

The main theorem of Hilbert spaces is the Hilbert projection theorem.

Consequences:

Thm: Let V be a Hilbert space. Then

a) If $W \subseteq V$ is any subspace, then $W^\perp$ is closed. Furthermore, $W^{\perp\perp} = \overline{W}$.

In particular, $W^{\perp\perp} = W$ iff W is closed.

b) If W is a closed subspace, then $W + W^\perp = V$.

In particular, $V = W \oplus W^\perp$ is an orthogonal decomposition.

Remark: These theorems can fail if V is not Hilbert.

Lemma: For any inner product space V, if we fix a vector $v \in V$, then the function $V \to \mathbb{C}$

$$w \mapsto \langle v, w \rangle$$

is a continuous function.

This follows from the following relationship expressing the inner product in terms of the norm

Polarization identity

Set $R(v,w) = \frac{1}{2}\left(\|v\|^2 + \|w\|^2 - \|v-w\|^2\right)$

Then $\langle v, w \rangle = R(v,w) + i\,R(iv,w)$.

We can use the lemma to prove that if $W \subseteq V$

is a subspace, then $W^\perp$ is closed.

It is easy to see that $\bigoplus_{n \in \mathbb{N}} \mathbb{C}$ is not Banach.

Consider the series $\sum_{n=0}^{\infty} \frac{1}{2^n} |n\rangle$.

This absolutely converges since $\sum_{n=0}^{\infty} \| \frac{1}{2^n} |n\rangle \|$
$= \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$

But the series does not converge, since it would have to converge to something like $v = \sum_{n=0}^{N} a_n |n\rangle$. But then the difference

between $\sum_{n=0}^{k} \frac{1}{2^n} |n\rangle$ and v must have norm at least $\frac{1}{2^{N+1}}$ for all $k > N$.

So $\bigoplus_{n\in\mathbb{N}} \mathbb{C}$ is not Banach.

Note that $\bigoplus_{n\in\mathbb{N}} \mathbb{C} = \{ f : \mathbb{N} \to \mathbb{C} \mid f(n) = 0 \}$
for all
but finitely
many n

A possible Banach space would be

$$\prod_{n\in\mathbb{N}} \mathbb{C} = \{ f : \mathbb{N} \to \mathbb{C} \}.$$

But then we get things like $(1, 1, 1, \ldots)$
with norm $\sqrt{1^2 + 1^2 + 1^2 + \ldots} = \infty$.

What if we just take the vectors with finite norm?

Def: Let S be a subset of $\mathbb{Z}$.
Then define the space

$$\ell^2(S) \doteq \left\{ f : S \to \mathbb{C} \;\middle|\; \sum_{x \in S} |f(x)|^2 < \infty \right\}$$

This is called ℓ^2-space.

This has an inner product

$$\langle f_1, f_2 \rangle = \sum_{x \in S} f_1(x)^* f_2(x) .$$

If S is finite, the norm condition is vacuous

so $\ell^2(S) = \{ f : S \to \mathbb{C} \} = \bigoplus_{x \in S} \mathbb{C}$ is finite dimensional.

Def: A dense spanning set in a normed space is a set whose span is dense.

A dense spanning orthonormal set in an inner prod space is called a (dense) orthonormal basis.

Dense orthonormal bases are not bases in general!

The usual sort of basis will be emphasized by calling it a linear basis.

Def: Given $S \subseteq \mathbb{Z}$ and a collection of Hilbert spaces $\{V_x\}_{x \in S}$ indexed by S, the Hilbert direct sum or completed direct sum is

$$\widehat{\bigoplus_{x \in S}} V_x \doteq \left\{ f : S \to \bigcup_{x \in S} V_x \;\middle|\; f(x) \in V_x \ \forall x \in S, \ \sum_{x \in S} \|f(x)\|^2 < \infty \right\}.$$

We have the containments

$$\bigoplus_{x \in S} V_x \subseteq \widehat{\bigoplus_{x \in S}} V_x \subseteq \prod_{x \in S} V_x$$

As an example, $\ell^2(S) = \widehat{\bigoplus_{x \in S}} \mathbb{C}$.

To define more Hilbert spaces, we want the index set S to be a topological space like $\mathbb{R}$ or S^1.

In that case, we'll define

$$L^2(S) = \int_S^{\wedge} \mathbb{C} \, d_x,$$

the direct integral of Hilbert spaces.

To make sense of this we need integration of both vectors and vector spaces themselves. There is a great framework for doing these things, using measure theory.